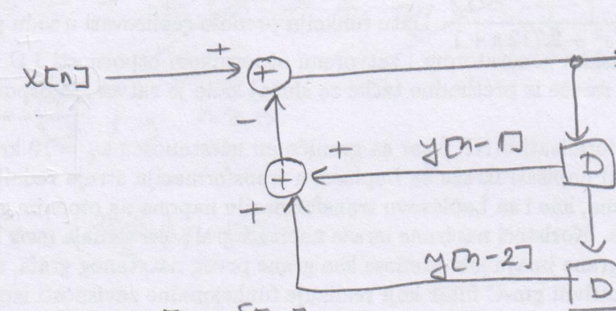
① $y[n] + y[n-1] + y[n-2] = x[n]$

a) $y[n] = -y[n-1] - y[n-2] + x[n]$



b) $h_1[n] + h_1[n-1] + h_1[n-2] = \delta[n]$

$h_1[0] = 1 \quad , \quad h_1[1] = -h_1[0] = -1$

$P(\lambda) = \lambda^2 + \lambda + 1$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm j\sqrt{3}}{2} \Rightarrow \text{II квадрант}$$

$\rho = 1 \quad , \quad \varphi = \frac{2\pi}{3}$

$\operatorname{arctg}(\varphi) = -\sqrt{3}$

$h[n] = \left(A \cdot \sin(n\varphi) + B\cos(n\varphi)\right)\mu[n]$

$h[0] = B = 1$

$$h[1] = A \cdot \underbrace{\sin\left(\frac{2\pi}{3}\right)}_{+\sqrt{3}/2} + B \cdot \underbrace{\cos\left(\frac{2\pi}{3}\right)}_{-1/2} = -1$$

$$A \cdot \frac{\sqrt{3}}{\cancel{2}} = -\frac{1}{\cancel{2}} \Rightarrow A = -\frac{\sqrt{3}}{3}$$

$$h[n] = \left(-\frac{\sqrt{3}}{3}\sin\left(\frac{2\pi}{3}n\right) + \cos\left(\frac{2\pi n}{3}\right)\right)\mu[n]$$

c) $$y_{p1}[n] = h_1[n] * x[n] = \sum_{m=-\infty}^{+\infty}\left(A\sin(m\varphi) + B\cos(m\varphi)\right)\cdot\mu[m]\cdot\mu[n-m] =$$

$$= \mu[n]\cdot\left(-A\cdot\frac{\cos(\varphi(m-\frac{1}{2}))}{2\sin\frac{\varphi}{2}} + B\cdot\frac{\sin(\varphi\cdot(m-\frac{1}{2}))}{2\sin\frac{\varphi}{2}}\right)\Bigg|_0^{n+1} =$$

$$= \mu[n]\cdot\left(\frac{\frac{\sqrt{3}}{3}\left(\cos\left((n+\frac{1}{2})\cdot\frac{2\pi}{3}\right) - \cancel{\cos\frac{\pi}{3}}^{1/2}\right) + \sin\left((n+\frac{1}{2})\cdot\frac{2\pi}{3}\right) + \cancel{\sin\frac{\pi}{3}}^{\sqrt{3}/2}}{\cancel{2\sin\frac{\pi}{3}}_{\frac{\sqrt{3}}{2}}}\right) =$$

$$= \mu[n]\cdot\left(\frac{1}{3} + \frac{1}{3}\cos\left((n+\frac{1}{2})\cdot\frac{2\pi}{3}\right) + \frac{1}{\sqrt{3}}\sin\left((n+\frac{1}{2})\frac{2\pi}{3}\right)\right)$$

$y_p[n] = y_{p1}[n] - y_{p1}[n-1]$

$y_s[n] = A\sin(n\varphi) + B\cos(n\varphi)$

$y_s[0] = B = 1$

$$y_s[1] = 2 = A\cdot\frac{\sqrt{3}}{2} - \frac{B}{2} \Rightarrow \frac{A\sqrt{3}}{\cancel{2}} = \frac{5}{\cancel{2}}$$

$$A = \frac{5}{\sqrt{3}}$$

$$\cos\left(\left(n+\frac{1}{2}\right)\frac{2\pi}{3}\right) = \cos\frac{n2\pi}{3}\cdot\cos\frac{\pi}{3} - \sin\frac{2n\pi}{3}\cdot\sin\frac{\pi}{3}$$

$$\sin\left(\left(n+\frac{1}{2}\right)\frac{2\pi}{3}\right) = \sin\frac{2n\pi}{3}\cdot\cos\frac{\pi}{3} + \cos\frac{2n\pi}{3}\cdot\sin\frac{\pi}{3}$$

$$y_p[n] = \frac{1}{3} + \frac{1}{6}\cdot\cos\frac{2n\pi}{3} \cancel{- \frac{\sqrt{3}}{6}\sin\frac{2n\pi}{3}} \cancel{+ \frac{\sqrt{3}}{6}\sin\left(\frac{2\pi n}{3}\right)} + \frac{\cancel{\sqrt{3}}}{3}\cdot\frac{\cancel{\sqrt{3}}}{2}\cos\left(\frac{2n\pi}{3}\right) =$$

$$= \frac{1}{3} + \frac{\cancel{4}^{\,2}}{\cancel{6}_{\,3}}\cos\left(\frac{2n\pi}{3}\right), \quad n \geq 0$$

$$y_p[n] = \left(\frac{1}{3} + \frac{2}{3}\cos\left(\frac{2n\pi}{3}\right)\right)u[n]$$

$$y_s[n] = \left(\frac{5}{\sqrt{3}}\sin\left(\frac{2n\pi}{3}\right) + \cos\left(\frac{2n\pi}{3}\right)\right)u[n]$$

$$y[n] = \left(\frac{1}{3} + \frac{5}{3}\cos\left(\frac{2n\pi}{3}\right) + \frac{5}{\sqrt{3}}\sin\left(\frac{2n\pi}{3}\right)\right), \quad n \geq 0$$

d) $(E^2+E+1)\, y_{pr}[n-2] = x[n]$

$$y_{pr}[n-2] = \frac{E1}{E^2+E+1} = \frac{1}{P(1)} = \frac{1}{3}$$

$$y_{pr}[n] = \frac{1}{3}\, u[n]$$

$$y_h[n] = A\sin(n\varphi) + B\cos(n\varphi)$$

из тачке c) је: $y[0] = y_s[0] + y_p[0] = 1 + 1 = 2$

$$y_p[0] = \frac{1}{3} + \frac{1}{3}\underbrace{\cos\left(\frac{\pi}{3}\right)}_{1/2} + \frac{1}{\sqrt{3}}\underbrace{\sin\left(\frac{\pi}{3}\right)}_{\sqrt{3}/2} = \frac{1}{3} + \frac{1}{6} + \frac{1}{2} = 1$$

$$y[1] = \underbrace{y_s[1]}_{2} + y_p[1] = 2$$

$y_p[1] = -y_p[0] + x[1] = 0$ (на основу формуле)

$$y_p[1] = \frac{1}{3} + \frac{1}{3}\cdot\underbrace{\cos\left(\frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{2}\pi}{\cancel{3}}\right)}_{-1} + \frac{1}{\sqrt{3}}\underbrace{\sin\left(\frac{3}{2}\cdot\frac{2\pi}{3}\right)}_{0} = 0$$

$$y[n] = y_{pr}[n] + y_h[n] = \frac{1}{3} + A\sin\left(\frac{2\pi n}{3}\right) + B\cos\left(\frac{2\pi n}{3}\right)$$

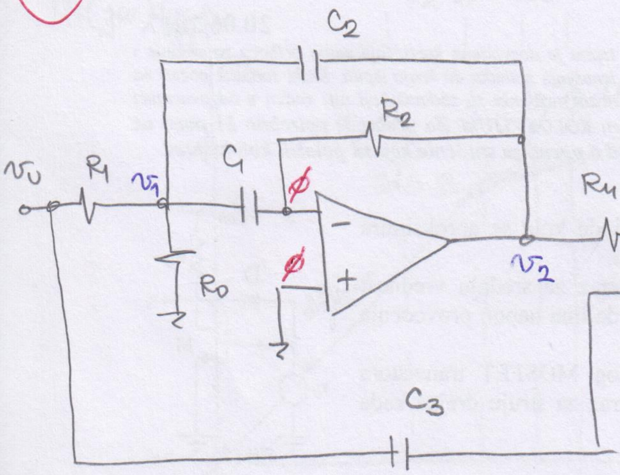
$$y[0] = \frac{1}{3} + B = 2 \Rightarrow B = \frac{5}{3}$$

$$y[1] = \frac{1}{3} + A\cdot\frac{\sqrt{3}}{2} + B\cdot\left(-\frac{1}{2}\right) = \frac{1}{3} - \frac{5}{6} + A\frac{\sqrt{3}}{2} = 2$$

$$A\frac{\sqrt{3}}{2} = 2 + \frac{\cancel{3}^{1}}{\cancel{6}_{2}} = \frac{5}{2}$$

$$A = \frac{5}{\sqrt{3}}$$

3



$$R_0 = \frac{R}{10} R$$

$$R_1 = R_2 = R_4 = R_5 = R$$

$$C_1 = C_2 = C$$

$$C_3 = \frac{C}{2}$$

$$v_I = -\frac{\cancel{R_5}}{\cancel{R_4}} v_2 - sC_3 \cancel{R_5} v_U$$

(R above the crossed R_4 replaced by 1; R beneath the crossed R_5)

$$v_2 = -v_I - sC_3 R v_U$$

$$v_2 = -R_2 \cdot sC_1 v_1 = -sCR v_1$$

$$v_1 = \frac{v_I + sC_3 R v_U}{sCR} = \frac{v_I}{sCR} + \frac{v_U}{2}$$

$$\frac{v_1 - v_U}{R_1} + \frac{v_1}{R_0} + sC_1 v_1 + sC_2 (v_1 - v_2) = 0$$

$$\frac{v_U}{R} = \left(\frac{1}{R} + \frac{1}{R_0} + sC \cdot 2\right) \cdot \frac{v_I + sC_3 R v_U}{sCR} + sC \cdot (v_I + sC_3 R v_U) =$$

$$= v_I \cdot \frac{\frac{1}{R} + \frac{1}{R_0} + 2sC + (sC)^2 R}{sCR} + \frac{sC_3 R \left(\frac{1}{R} + \frac{1}{R_0} + 2sC + (sC)^2 \cdot R\right)}{sCR} v_U \quad / \cdot sCR$$

$$v_U \cdot \left(\left(-\frac{1}{R} - \frac{1}{R_0} - 2sC - (sC)^2 R\right) \cdot sC_3 R + sC\right) = v_I \left(\frac{1}{R} + \frac{1}{R_0} + 2sC + (sC)^2 R\right)$$

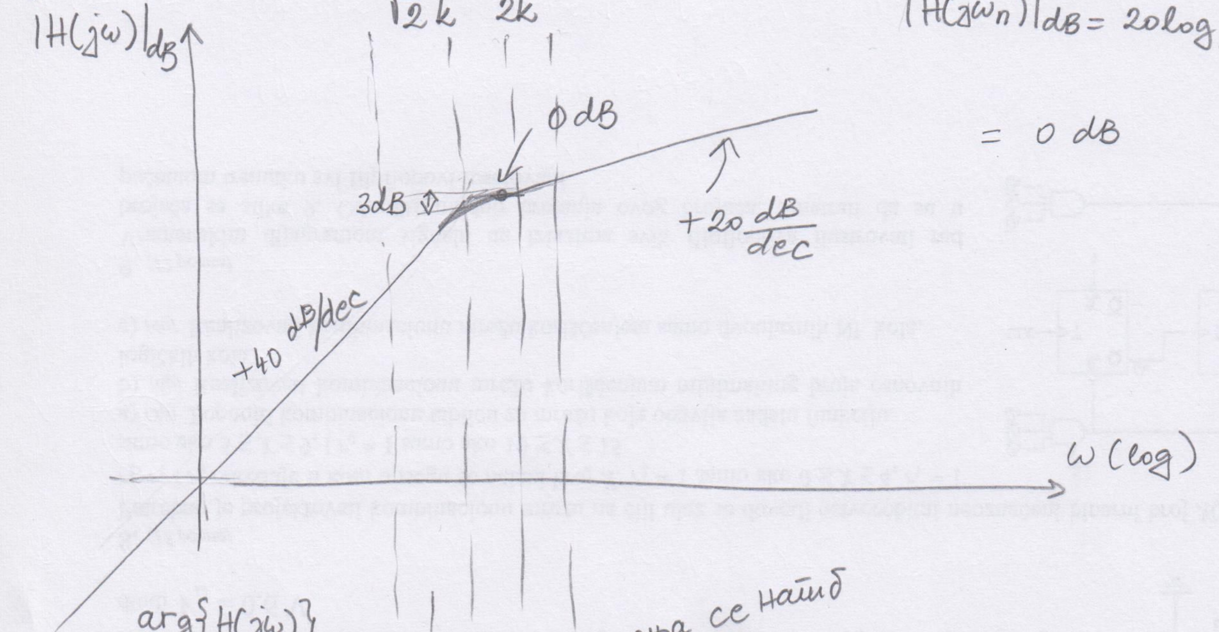
$$s\left(C - C_3 \cancel{R} \cdot \frac{R + R_0}{\cancel{R} R_0}\right) = s \cdot \left(C - C_3 \frac{R + R_0}{R_0}\right) = 0$$

$$-v_U \cdot \left(2sC \cdot \frac{sCR}{2} + (sC)^2 R \cdot \frac{sCR}{2}\right) = v_I \left(\frac{2}{R} + 2sC + (sC)^2 R\right)$$

$$H(s) = -\frac{\frac{(sCR)^3}{2} + (sCR)^2}{(sCR)^2 + 2sCR + 2} = -\frac{1}{2} \frac{CR\, s^3 + 2s^2}{s^2 + \frac{2s}{CR} + \frac{2}{(CR)^2}} = -\frac{1}{2} \frac{CR\, s^2 \left(s + \frac{2}{CR}\right)}{s^2 + \frac{2s}{CR} + \frac{2}{(CR)^2}}$$

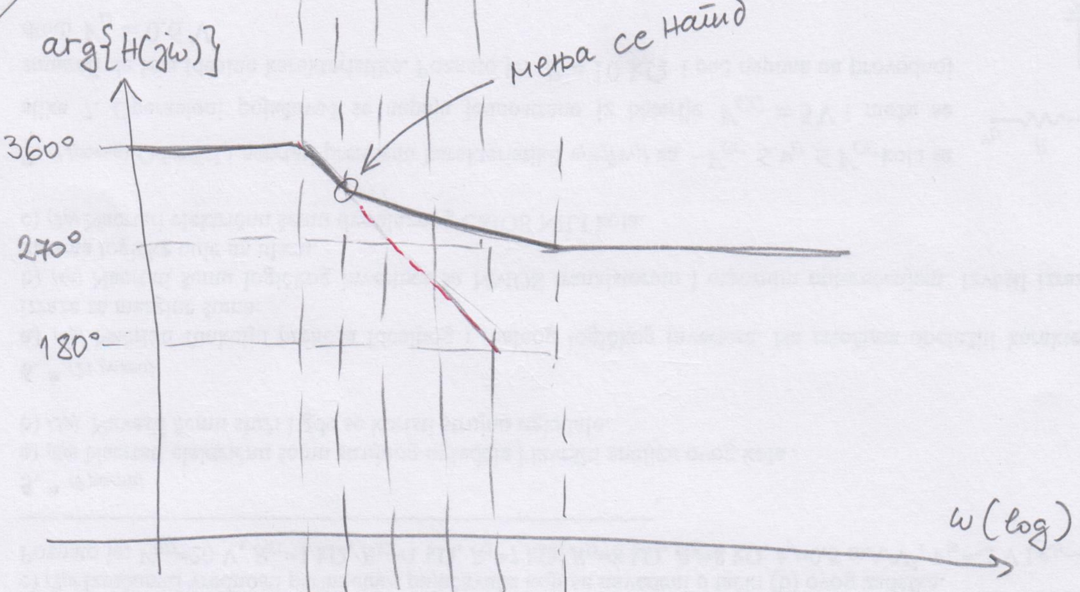
$$= -\frac{10^{-3}}{2} \frac{s^2 (s + 2000)}{(s^2 + 2000s + 2 \cdot 10^6)}$$

б)



$$|H(j\omega_n)|_{dB} = 20\log_{10}\left|-5\cdot10^4\frac{\cancel{\omega_n^2}\cdot\omega_n}{\cancel{\omega_n^2}}\right| =$$

$$= 0\ dB$$

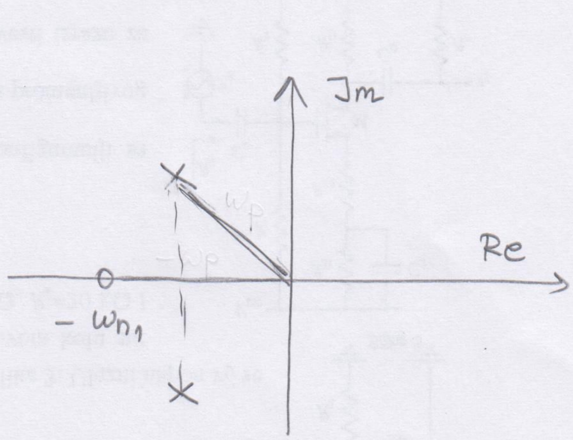


$$H(s) = -\frac{10^{-3}}{2}\cdot\frac{(s+\omega_{n1})\cdot s^2}{\left(s^2+\frac{s\omega_p}{Q}+\omega_p^2\right)}$$

$$\omega_{n1} = 2000\ \frac{rad}{s}$$

$$\omega_p = \sqrt{2}\cdot10^3\ \frac{rad}{s}$$

$$Q = \frac{\sqrt{2}}{2}$$



c) Коло може да ради као пропусник високих учестаности.

3. Kontinualni LTI system je opisan diferencijalnom jednačinom: $(D^2-1)y(t)=(D+2)x(t)$. Rešavanjem u vremenskom domenu

a) [10] odrediti impulsni odziv sistema.

b) [20] Odrediti sopstveni odziv sistema ako je $x(t)=(\cos 3t+\sin 3t)\mathrm{u}(t)$

Resenje:

Kada je red polinoma *P* veci od reda polinoma *Q* tada ne mogu da se dobiju impulsi u prinudnom odzivu ($y(t)$ ne sadrzi $\delta(t)$ i njegove izvode)

Primenom $Q(D)$ na $x(t)$ dobija se $f(t)= f_1(t)+ f_2(t)$ pri cemu $f_1(t)$ ne sadrzi impulse a $f_2(t)$ sadrzi samo impulse. Sopstveni odziv moze da se dobije superpozicijom na odvojeno $f_1(t)$ i $f_2(t)$. Odziv na $f_1(t)$ se moze dobiti operacionim racunom jer $f_1(t)$ ne sadrzi impulse (preinicijalni uslovi = postinicijalni uslovi =0) dok je odziv na $f_2(t)$ vec odredjen u tacki a)

2) a) $P(D)y(t)=Q(D)x(t)$ $\boxed{\text{RED}(P)>\text{RED}(Q) \Rightarrow \text{НЕМА } \delta \text{ у } y(t)}$

$s_1(t)=C_1e^{-t}+C_2e^{t}+\frac{1}{P(0)}=C_1e^{-t}+C_2e^{t}-1;\ t>0$

$s_1'(t)=-C_1e^{-t}+C_2e^{t};\ t>0$

$s_1(0^+)=s_1'(0^+)=0$

$t>0$

$C_1+C_2-1=0$ $\quad C_1+C_2=1$

$-C_1+C_2=0$ $\quad C_1-C_2=0$ $\Big\}\ C_1=C_2=\frac{1}{2}\Rightarrow \boxed{s_1(t)=\frac{1}{2}(e^{t}+e^{-t})-1}$

$\boxed{h_1(t)=s_1'(t)}=\boxed{\left(-\frac{1}{2}e^{-t}+\frac{1}{2}e^{t}\right)u(t)}$

$h(t)=(D+2)h_1(t)=\left(\frac{1}{2}e^{-t}+\frac{1}{2}e^{t}+e^{t}-e^{-t}\right)u(t)=\boxed{\left(-\frac{1}{2}e^{-t}+\frac{3}{2}e^{t}\right)u(t)}$

диференцирање $x(t)$

б) $f(t)=(D+2)x(t)$ $\left(\boxed{\text{ДЕСНА СТРАНА ЈЕДНАЧИНЕ САДРЖИ } \delta \quad y(0^-)\neq y(0^+)\ldots}\right)$

$f(t)=(-3\sin 3t+3\cos 3t)u(t)+$

$+(\overset{1}{\cos 3t}+\overset{0}{\sin 3t})\delta(t)+2(\cos 3t+\sin 3t)u(t)$

$f(t)=\underbrace{(5\cos 3t-\sin 3t)u(t)}_{f_1(t)}+\underbrace{\delta(t)}_{f_2(t)}=f_1(t)+f_2(t)$

$P(D)y(t)=f_1(t)+f_2(t)$ $\quad y(t)=y_1(t)+y_2(t)$ СУПЕРПОЗИЦИЈА

$P(D)\ y_1(t)=f_1(t)$ НЕМА

$P(D)\ y_2(t)=f_2(t)$ ИМА $\Big\}$ δ СА ДЕСНЕ СТРАНЕ

$\boxed{y_2(t)=h_1(t)}$ $\quad \omega=3;\ \boxed{-\omega^2=-9}$

$y_1(t)=C_1e^{-t}+C_2e^{t}+\frac{1}{P(D)}(5\cos 3t-\sin 3t);\ t>0$

$y_1(t)=C_1e^{-t}+C_2e^{t}+\frac{1}{-9-1}(5\cos 3t-\sin 3t)$

$y_1(t)=C_1e^{-t}+C_2e^{t}-\frac{1}{10}(5\cos 3t-\sin 3t)$

$y_1'(t)=-C_1e^{-t}+C_2e^{t}-\frac{1}{10}(-15\sin 3t-3\cos 3t)$

$y_1(0^+)=0$ $\quad C_1+C_2-\frac{1}{10}\cdot 5=0$

$y_1'(0^+)=0$ $\quad -C_1+C_2+\frac{1}{10}\cdot 3=0$ $\quad \boxed{C_1=+\frac{4}{10}\quad C_2=+\frac{1}{10}}$

$$y(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} + \frac{4}{10}e^{-t} + \frac{1}{10}e^{+t} - \frac{1}{10}(5\cos 3t - \sin 3t) \quad t > 0$$

$$y(t) = -\frac{1}{10}e^{-t} + \frac{6}{10}e^{t} - \frac{1}{10}(5\cos 3t - \sin 3t) \quad t > 0$$

Drugi nacin:

Ako je odziv na $x(t)$ jednak $y_1(t)$, tada je odziv na $Q(D)\,x(t)$ jednak $Q(D)\,y_1(t)$

$x(t)$ ne sadrzi impulse pa moze da se resi operacionim racunom

$$y_1(t) = c_1 e^{-t} + c_2 e^{t} - \frac{1}{10}(\cos 3t + \sin 3t) \quad t > 0$$

$$y_1'(t) = -c_1 e^{-t} + c_2 e^{t} - \frac{1}{10}(-3\sin 3t + 3\cos 3t)$$

$$y_1(0^+) = 0 \Rightarrow c_1 + c_2 - \frac{1}{10} = 0 \qquad c_1 = -\frac{1}{10}$$

$$y_1'(0^+) = 0 \Rightarrow -c_1 + c_2 - \frac{1}{10}\cdot 3 = 0 \qquad c_2 = \frac{2}{10}$$

$$y_1(t) = -\frac{1}{10}e^{-t} + \frac{2}{10}e^{t} - \frac{1}{10}(\cos 3t + \sin 3t)$$

$$y_1'(t) = \frac{1}{10}e^{-t} + \frac{2}{10}e^{t} - \frac{3}{10}(\cos 3t - \sin 3t)$$

$$y(t) = y_1'(t) + 2y_1(t) =$$

$$= \frac{1}{10}e^{-t} - \frac{2}{10}e^{-t} + \frac{2}{10}e^{t} + \frac{4}{10}e^{t} - \frac{1}{10}(5\cos 3t + \sin 3t)$$

$$= -\frac{1}{10}e^{-t} + \frac{6}{10}e^{t} - \frac{1}{10}(5\cos 3t - \sin 3t) \; ; \; t > 0$$

Komentra za konvoluciju

Direktna primena konvolucije i parcijalna integracija, su u ovakvom slučaju mukotrpan posao podložan greskama pa ga ne treba koristiti

Po nekad je moguće rešiti konvolucioni integral bez parcijalne integracije: U ovom slucaju ako se nadje odziv na kompleksnu eksponencijalnu pobudu e^{j3t}, pa se u resenju odvoje imaginarni i realni deo, dobice se odzivi na sinus i kosinus

$g(t) = x_1(t) + j x_2(t)$

$g(t) = e^{j3t}$

$g(t) * h(t) \Rightarrow$

ОДЗИВ НА $g(t)$ ЈЕ $\in Z$
ОДЗИВ НА $x(t) = x_1(t) + x_2(t)$
ЈЕ ЗБИР РЕАЛНОГ И ИМАГ. ДЕЛА
ОДЗИВА НА $g(t)$!

$$e^{t} * e^{j3t} = e^{at} * e^{bt} = \frac{e^{t} - e^{j3t}}{1-3j}$$

$$e^{-t} * e^{j3t} = e^{-at} * e^{bt} = -\frac{e^{-t} - e^{j3t}}{1+3j}$$

$$\frac{1}{2}\left(\frac{e^{-t} - e^{j3t}}{1+3j}\right) + \frac{3}{2}\left(\frac{e^{t} - e^{j3t}}{1-3j}\right) =$$

$$= \frac{1}{2}\cdot\frac{1}{10}\left(\left(e^{-t} - \cos 3t - j\sin 3t\right)(1-3j) + 3\left(e^{t} - \cos 3t - j\sin 3t\right)(1+3j)\right) =$$

$$= \frac{1}{2}\cdot\frac{1}{10}\Big(\left(e^{-t} - \cos 3t - j\sin 3t - 3je^{-t} + 3j\cos 3t - 3\sin 3t\right) +$$

$$+ 3\left(e^{t} - \cos 3t - j\sin 3t + 3je^{t} - 3j\cos 3t + 3\sin 3t\right)\Big)$$

РАЗДВОЈИ СЕ ИМАГИНАРНИ И РЕАЛНИ ДЕО
А ЗАТИМ СЕ САБЕРУ

$$y(t) = -\frac{1}{10}e^{-t} + \frac{6}{10}e^{t} - \frac{1}{10}\left(5\cos 3t + \sin 3t\right)$$

4.

a) [5] Odrediti Furijeovu transformaciju signala $x_F(t) = \sin(\pi t/2)\big(\mathrm{u}(t) - \mathrm{u}(t-1)\big)$

b) [5] Odrediti koeficijente razvoja u kompleksni Furijeov red signala kome je osnovna perioda trajanja 2 i koji je na prvoj osnovnoj periodi jednak signalu iz tačke a)

4) a) $u(t) - u(t-1) = \mathrm{rect}\left(t - \frac{1}{2}\right)$

$$\mathcal{F}\left\{\mathrm{rect}\left(t - \tfrac{1}{2}\right)\right\} = \mathrm{sinc}\frac{\omega}{2\pi}\, e^{-j\frac{\omega}{2}}$$

$$X_F(j\omega) = \frac{1}{2\pi}\,\mathrm{sinc}\frac{\omega}{2\pi}\, e^{-\frac{j\omega}{2}} * j\pi\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right) \qquad \omega_0 = \frac{\pi}{2}$$

$$X_F(j\omega) = \frac{j}{2}\left(\mathrm{sinc}\frac{\omega+\omega_0}{2\pi}\, e^{-j\frac{(\omega+\omega_0)}{2}} - \mathrm{sinc}\frac{\omega-\omega_0}{2\pi}\, e^{-\frac{j(\omega-\omega_0)}{2}}\right)$$

b) $X[k] = \frac{1}{T_F} X_F(j\omega)\Big|_{\omega = k\omega_F}$ $\qquad T_F = 2$

$$\omega_F = \frac{2\pi}{T_F} = \pi$$

$$X[k] = \frac{j}{4}\left(\mathrm{sinc}\frac{k\pi + \frac{\pi}{2}}{2\pi}\, e^{-\frac{j\left(k\pi + \frac{\pi}{2}\right)}{2}} - \mathrm{sinc}\frac{k\pi - \frac{\pi}{2}}{2\pi}\, e^{-\frac{j\left(k\pi - \frac{\pi}{2}\right)}{2}}\right)$$